Inequality with constraint that involve cubic root.
https://www.linkedin.com/feed/update/urn:li:activity:6786997320488693761
Let a,b,c be positive real numbers such that $a+b+c=4\sqrt[3]{abc}$.
Prove that $2(ab+bc+ca)+4\min\{a^2,b^2,c^2\}\geq a^2+b^2+c^2$.
Solution by Arkady Alt, San Jose, California, USA.
Since $a+b+c=4\sqrt[3]{abc}$ then $2(ab+bc+ca)+4\min\{a^2,b^2,c^2\}\geq a^2+b^2+c^2 \Leftrightarrow$
$4(ab+bc+ca)+4\min\{a^2,b^2,c^2\}\geq(a+b+c)^2=16\sqrt[3]{a^2b^2c^2} \Leftrightarrow$
(**1**) $ab+bc+ca+\min\{a^2,b^2,c^2\}\geq 4\sqrt[3]{a^2b^2c^2}$.
Due homogeneity and symmetry of the inequality and the constraint, we can assume that $c=\min\{a,b,c\}=1$. Then constraint and inequality (**1**) becomes, respectively, $a+b+1=4\sqrt[3]{ab}$ and $ab+b+a+1\geq 4\sqrt[3]{a^2b^2}$.
We have $ab+b+a+1-4\sqrt[3]{a^2b^2}=ab+4\sqrt[3]{ab}-4\sqrt[3]{a^2b^2}=\sqrt[3]{ab}\left(\sqrt[3]{a^2b^2}-4\sqrt[3]{ab}+4\right)=$
$\sqrt[3]{ab}\left(\sqrt[3]{ab}-2\right)^2\geq 0.$